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The Oscillation Theorem for Tchebycheff Spaces of Bounded Functions, and a Converse

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A $(k+1)$-dimensional vector space U of real-valued functions defined on a subset of the real line is a Tchebycheff space (the linear space generated by a Tchebycheff system) if the number of zeros and the number of alternations in sign of each nonzero element of U is at most k. We show here that if U is a Tchebycheff space of bounded functions defined on a subset T of the real line, then for any pair of real-valued functions h_0, h_1 defined on T for which there is an element of U lying between h_0 and h_1 and bounded away from them, there exists an element of U that lies between h_0 and h_1 and oscillates between them exactly k times. Additionally, a converse is given.

1. INTRODUCTION

Suppose U is a Tchebycheff space (see [2]) of bounded functions defined on a subset T of the real line and suppose h_0, h_1 are two arbitrary real-valued functions defined on T such that for some $p \in U$ and $\epsilon > 0$,

$$h_0(t) + \epsilon \leqslant p(t) \leqslant h_1(t) - \epsilon \qquad (+)$$

for all $t \in T$. We prove here that there is a $\underline{u} \in U$ such that $h_0(t) \leqslant \underline{u}(t) \leqslant h_1(t)$ for all $t \in T$ and $\underline{u}$ oscillates k times between h_0 and h_1, touching each alternately, where k is the degree of U.

This theorem, which we refer to as the "oscillation theorem for T-spaces of bounded functions," has a heritage in a series of representation theorems which go back to the well-known theorem of Pólya and Szegö [8] that a real polynomial h, nonnegative on the entire real line, can be expressed as

$$h(t) = (A(t))^2 + (B(t))^2,$$

where A and B are real polynomials whose respective degrees do not exceed half the degree of h. This theorem was later refined to allow for h to be nonnegative simply for $t \geqslant 0$. In this case, h can be expressed as

$$h(t) = (A(t))^2 + (B(t))^2 + t[(C(t))^2 + (D(t))^2],$$

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where A and B are as before and C and D are real polynomials whose respective degrees do not exceed $\frac{1}{2}(\deg h - 1)$. Attributed to M. Fekete is that when $h(t)$ is nonnegative simply for $-1 \leqslant t \leqslant 1$, h can be expressed as

$$h(t) = (A(t))^2 + (1 - t^2)(B(t))^2,$$

where $\deg A = \deg B + 1 = \frac{1}{2} \deg h$, and this was refined by the following result attributed to F. Luckács.

Let $h(t)$ be a real polynomial of degree k, nonnegative for $-1 \leqslant t \leqslant 1$. Then h can be expressed as

$$\begin{aligned} h(t) &= (A(t))^2 + (1 - t^2)(B(t))^2 && \text{if } k \text{ is even,} \\ &= (1 + t)(C(t))^2 + (1 - t)(D(t))^2 && \text{if } k \text{ is odd,} \end{aligned} \tag{L}$$

where A, B, C, D are real polynomials whose degrees do not exceed $k/2$, $(k/2) - 1$, $(k - 1)/2$, and $(k - 1)/2$, respectively.

These four results appear as problems 44–47 in [8, VI, Sect. 6, p. 82] (solutions on pp. 275–276). See also [9, pp. 4–5]. Representation (L) follows from the theorem of Fejér [1], which gives a nonnegative trigonometric polynomial h with real coefficients as the square of the modulus of an algebraic polynomial p of the same degree: $h(\theta) = |p(z)|^2$ for $z = e^{i\theta}$. However, the representation of h in terms of p is not unique and thus representation (L) is not unique.

In 1953, Karlin and Shapley [5, p. 35] showed that in representation (L), if h has fewer than k zeros counting multiplicities in $[-1, 1]$, then A, B, C, and D could be required to have respective degrees precisely $k/2$, $(k/2) - 1$, $(k - 1)/2$, and $(k - 1)/2$, and in addition all their roots could be required to be real and to lie in the interval $[-1, 1]$. In this case, the two polynomials $\underline{u}$, $\bar{u}$ defined as

$$\begin{aligned} &\bar{u}(t) = (A(t))^2, && \underline{u}(t) = (1 - t^2)(B(t))^2 && \text{when } k \text{ is even,} \\ &\underline{u}(t) = (1 + t)(C(t))^2, && \bar{u}(t) = (1 - t)(D(t))^2 && \text{when } k \text{ is odd,} \end{aligned}$$

each oscillate between 0 and $h(t)$ exactly k times. Specifically, they showed that there are two polynomials $\underline{u}, \bar{u}$ and $k + 1$ points t_i satisfying $-1 = t_0 < t_1 \leqslant \cdots \leqslant t_{k-1} \leqslant t_k = 1$ such that

$$\begin{aligned} 0 \leqslant \underline{u}(t) &\leqslant h(t) \qquad \text{for} \quad t \in [-1, 1], \\ \underline{u}(t_0) = (h - \underline{u})(t_1) = \underline{u}(t_2) &= (h - \underline{u})(t_3) = \cdots = 0; \\ 0 \leqslant \bar{u}(t) &\leqslant h(t) \qquad \text{for} \quad t \in [-1, 1], \\ (h - \bar{u})(t_0) = \bar{u}(t_1) = (h - \bar{u})(t_2) &= \bar{u}(t_3) = \cdots = 0. \end{aligned} \tag{osc}$$

The even-indexed t_i's interior to $[-1, 1]$ must be double zeros of $\underline{u}$; and similarly for the odd-indexed t_i's and $\bar{u}$. For any $\underline{u}$ satisfying (osc), $h - \underline{u}$

must satisfy the role of $\bar{u}$ in (osc) (and conversely). Hence, once a polynomial $\underline{u}$ is found which satisfies (osc), $\bar{u}$ is determined and

$$h = \underline{u} + \bar{u}.$$

A simple counting argument applied to (osc) involving degree shows that $\underline{u}$ and $\bar{u}$ must each be unique (if existent) for any continuous function h such that $h(t) > 0$.

As it happens, the existence of polynomials $\underline{u}$, $\bar{u}$ satisfying (osc) (allowing different t_i's for each) does not depend upon h being a polynomial. Of course, $\underline{u} + \bar{u}$ is a polynomial of degree $\leqslant k$ even if h is not, and hence the representation $h = \underline{u} + \bar{u}$ is valid if and only if h is also a polynomial of degree $\leqslant k$. In 1963, Karlin [4] showed that if h is any positive continuous function and k is any positive integer, there exist two polynomials $\underline{u}$, $\bar{u}$ of degree k and $2(k + 1)$ points s_i, $t_i \in [-1, 1]$ such that

$$0 \leqslant \underline{u}(t), \bar{u}(t) \leqslant h(t) \qquad \text{for} \quad t \in [-1, 1], \tag{osc$'$}$$

$$t_i < t_{i+1}; \qquad \underline{u}(t_0) = (h - \underline{u})(t_1) = \underline{u}(t_2) = (h - \underline{u})(t_3) = \cdots = 0$$

$$s_i < s_{i+1}; \quad (h - \bar{u})(s_0) = \bar{u}(s_1) = (h - \bar{u})(s_2) = \bar{u}(s_3) = \cdots = 0.$$

This proof depended upon the compactness of $[-1, 1]$ (which could be replaced by any closed interval) and the continuity of h and polynomials, using as it did Brouwer's fixed-point theorem. In fact, there was no need for $\underline{u}$ and $\bar{u}$ actually to *be* polynomials, so long as they behaved reasonably well like polynomials. By applying a smoothing process to k-differentiable functions, an argument similar to that for polynomials showed that if h is any positive continuous function and U is any T-space of degree k of continuous functions (of which the polynomials of degree $\leqslant k$ are an example), then there exist $\underline{u}, \bar{u} \in U$ satisfying (osc$'$).

The final form of this theorem to date appeared in [6], where the authors show that if U is a T-space of continuous functions defined on some closed interval $[a, b]$ and if h_0 and h_1 are arbitrary continuous functions on $[a, b]$ such that for some $p \in U$, (+) is satisfied, then there exist $\underline{u}, \bar{u} \in U$ and $2(k + 1)$ points s_i, $t_i \in [a, b]$ such that (with $h_i = h_0$) when i is even and $h_i = h_1$ when i is odd) we have:

$$h_0(t) \leqslant \underline{u}(t), \qquad \bar{u}(t) \leqslant h_1(t) \qquad \text{for} \quad t \in [a, b] \text{ and for } i = 0, 1, \ldots, k,$$

$$t_i < t_{i+1}; \qquad (\underline{u} - h_i)(t_i) = 0; \tag{OSC}$$

$$s_i < s_{i+1}; \qquad (\bar{u} - h_{i+1})(s_i) = 0.$$

These two functions $\underline{u}$, $\bar{u}$ are unique, and if $h_0 = 0$ then $h_1 \in U$ if and only if $\bar{u} = h_1 - \underline{u}$, in which case $h_1 = \underline{u} + \bar{u}$.

In 1974 Pinkus [7] further extended this to allow h_0 and h_1 to be upper and lower semicontinuous, respectively. However, the continuity of the elements of U was still required.

In this paper we prove the corresponding theorem for arbitrary T-spaces of bounded function, wherein the interval $[a, b]$ is replaced by an arbitrary subset T of the real line and the elements of the T-space U need only be bounded (not necessarily continuous). The functions h_0 and h_1 can be completely arbitrary (so long as for some $p \in U$, $(*)$ is satisfied). Furthermore, this is the farthest that this line of theorems can be extended, as we show in Section 6.

Our proof derives from a new characterization of $\underline{u}$ as a solutoin to a pair of extremal problems which we informally describe next. Let us visualize the set of elements of U that lie between h_0 and h_1 as curves that start from the leftmost end of T and pass through the space between h_0 and h_1. Of these elements and for any $i \geq 0$, consider those which touch h_0 at the least possible value of the argument, say $t = r_0$, which next touch h_1 at the least possible value of $t \geq r_0$, say $t = r_1$, then next touch h_2 $(= h_0)$ at the least possible value of $t \geq r_1$, say $t = r_2$, and so on, finally touching h_{i+1} at the least possible value of $t \geq r_i$, say $t = r_{i+1}$. Of course for some i this set may be empty, in which case we set $r_j = +\infty$ for $j > i$. The elements of this subset of U, let us call it $U_{i+1} \subset U$, are those elements of U which oscillate as "fast" as possible between h_0 and h_1 in the interval $[r_0, r_{i+1}]$ (starting by touching h_0). This is the first extremal problem. The element $\underline{u}$ is one which then maximizes the oscillation in a different but related sense. Consider the smallest λ for which there is a set of $k - \lambda$ points ω_{ij} such that

$$\omega_{ij} \in]r_{i-1}, r_i[\cap T \qquad \text{and} \qquad \omega_{ij} < \omega_{ij+1}$$

for each $i = 0, 1, \ldots, \lambda - 1$ (with $r_{-1} = -\infty$). Define the linear form $\delta(u)$ for each $u \in U$ as

$$\delta(u) = \sum_{i=0}^{\lambda-1} (-1)^i \sum_j u(\omega_{ij}).$$

Among the elements of U that lie between h_0 and h_1 and touch h_i at r_i for $0 \leq i < \lambda$, the one that minimizes δ oscillates between h_0 and h_1 k times. This is $\underline{u}$.

To prove the theorem, the concept of "touching" in the previous sketch must be made precise. The complex variety of ways in which two discontinuous functions can "touch" one another greatly complicates the situation but, remarkably enough, the essence of the idea just sketched carries the theorem even in its most general case.

One complication is that unless the elements of U are continuous, we do not necessarily obtain $t_i < t_{i+1}$ as in (OSC), but rather $t_i \leq t_{i+1}$. This is because a discontinuous u can jump from h_0 to h_1 at a single point. While it is natural to consider such a jump as a valid term in an oscillation sequence, great care must be taken to avoid "invalid" oscillations. This is discussed forthwith in the beginning of the next section.

Before proceeding, we introduce some notation which is used throughout. The real line is denoted by $\mathbb{R}$, the set of positive integers by $\mathbb{N}$, the cardinality of a set S by card S, the closure of $T \subset \mathbb{R}$ by cl T. For any given $T \subset \mathbb{R}$, the set of real-valued functions on T is denoted by $\mathscr{F}(T)$. The bounded and continuous functions in $\mathscr{F}(T)$ are denoted respectively by $\mathscr{B}(T)$ and $\mathscr{C}(T)$.

Of course, $\mathscr{F}(T)$, $\mathscr{B}(T)$, and $\mathscr{C}(T)$ are all vector spaces over $\mathbb{R}$. Any vector space properties such as linear dependence or dimension, pertaining to elements or subsets $\mathscr{F}(T)$, are to be understood to be with respect to the real ground field.

The set $\mathscr{B}(T)$ is understood to be topologized by the sup norm: $\| u \| = \sup | u(t) |$ $(t \in T)$. Consistent with this, $\mathscr{F}(T)$ is topologized with the subbase: the sets

$$\{ g \in \mathscr{F}(T) \mid \| f - g \| < \epsilon \}$$

defined for all $f \in \mathscr{F}(T)$ and all $\epsilon > 0$. The topology for $\mathscr{F}(T)$ is all unions of finite intersections of elements from the subbase.

With respect to this topology, $\mathscr{F}(T)$ is Hausdorff and 1st-countable (each point has a countable neighborhood base). Thus a subspace $X \subset \mathscr{F}(T)$ is sequentially compact (every sequence in X admits a subsequence which converges to a point of X) iff every countable subset of X admits a limit point in X. And in either case X is closed.

2. Oscillation of a Function between Two Others

For $h_0, h_1 \in \mathscr{F}(T)$ we denote the set of functions that lie between h_0 and h_1 by $[h_0, h_1]$:

$$[h_0, h_1] = \{u \in \mathscr{F}(T) \mid \forall t \in T, h_0(t) \leqslant u(t) \leqslant h_1(t)\}.$$

The set of elements of $[h_0, h_1]$ that do not equal h_0 or h_1 anywhere is denoted by $]h_0, h_1[$:

$$]h_0, h_1[= \{u \in \mathscr{F}(T) \mid \forall t \in T, h_0(t) < u(t) < h_1(t)\}.$$

The set of functions in $]h_0, h_1[$ that are bounded away from h_0 and h_1 is denoted by $]]h_0, h_1[[$:

$$]]h_0, h_1[[= \{u \in \mathscr{F}(T) \mid \exists \epsilon > 0, \forall t \in T, h_0(t) + \epsilon \leqslant u(t) \leqslant h_1(t) - \epsilon\}.$$

When T is a compact set and the functions h_0, h_1 are continuous then

$$]h_0, h_1[\cap \mathscr{C}(T) =:]]h_0, h_1[[\cap \mathscr{C}(T). \tag{2.1}$$

For a continuous function $u \in [h_0, h_1] \cap \mathscr{C}(T)$, we say that u "oscillates" between h_0 and h_1 if u touches h_0 and h_1 alternatingly (see Fig. 1).

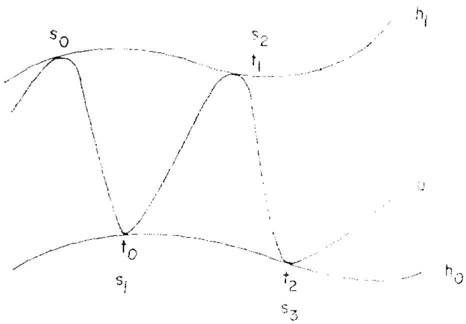

FIGURE 1

Since u cannot touch h_0 and h_1 at the same point, the points at which u touches h_0 and h_1 "alternatingly" is well defined in the natural way, and the number of such points gives a measure of the "oscillation" of u between h_0 and h_1. In general, without continuity, u can touch h_0 and h_1 at the same point and some of the ways in which this can happen are shown in Fig. 2.

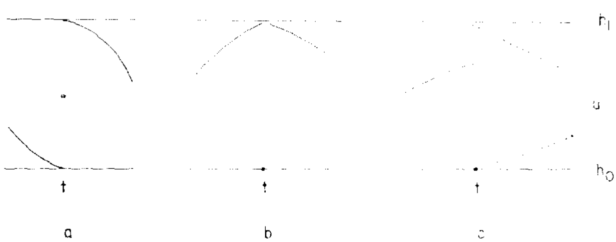

FIG. 2. Three ways of touching.

In order to distinguish between the various ways two functions may "touch" each other, we make the following definitions. For any $u, v \in \mathscr{F}(T)$, $t \in \operatorname{cl} T$, and $\mathbb{N}$ the positive integers, define

$$u \uparrow v : t \Leftrightarrow \forall n \in \mathbb{N}, \quad \exists t_n \in T, \quad t_n < t_{n+1},$$

such that

$$\lim_n t_n = t \qquad \text{and} \qquad \lim_n (u(t_n) - v(t_n)) = 0,$$

$$u \downarrow v : t \Leftrightarrow \forall n \in \mathbb{N}, \quad \exists t_n \in T, \quad t_{n+1} < t_n,$$

such that

$$\lim_n t_n = t \quad \text{and} \quad \lim_n (u(t_n) - v(t_n)) = 0,$$

$$u \uparrow\downarrow v : t \Leftrightarrow u \downarrow v : t \quad \text{and} \quad u \uparrow v : t,$$

$$u \sim v : t \Leftrightarrow u \downarrow v : t \quad \text{or} \quad u \uparrow v : t,$$

$$u = v : t \Leftrightarrow u(t) = v(t),$$

$$u \cong v : t \Leftrightarrow u \sim v : t \quad \text{or} \quad u(t) = v(t),$$

$$u \overset{\uparrow}{=} v : t \Leftrightarrow u \uparrow v : t \quad \text{or} \quad u(t) = v(t),$$

$$u \overset{\downarrow}{=} v : t \Leftrightarrow u \downarrow v : t \quad \text{or} \quad u(t) = v(t).$$

Note that $u \uparrow v : t \Leftrightarrow (u - v) \uparrow 0 : t$ and so on. Also $\not\uparrow$, $\not\downarrow$, $\not\cong$, $\not\sim$ are used to denote the negation of the respective symbols without the slashes. If $u \sim 0 : t$ we say that t is an *asymptotic zero* of u.

In Fig. 2, for example, $u \uparrow h_0 : t$ and $u \downarrow h_1 : t$ in case (a); in (c), $u \downarrow h_0 : t$ and $u \downarrow h_1 : t$; in (b), $u \downarrow\uparrow h_1 : t$ and $u(t) = h_0(t)$.

We now define oscillation sequences.

Let T be a bounded subset of $\mathbb{R}$ and let $h_0, h_1 \in \mathscr{F}(T)$ be such that $]]h_0, h_1[[\neq \varnothing$. Let $u \in [h_0, h_1]$. As in the preceding, henceforth set $h_j = h_0$ if j is even and $h_j = h_1$ if j is odd. The *lower oscillation sequence* of u relative to $[h_0, h_1]$ is $t_{-1}, t_0, t_1, t_2, \ldots$ defined recursively in terms of an auxiliary sequence $t'_{-1}, t_0', t_1', t_2', \ldots$ as follows.

Let $t'_{-1} = t_{-1} = -\infty$. For $j = 0, 1, \ldots$, if t_{j-1} has been defined, let

$$t_j' = \inf\{t > t_{j-1} \mid u \cong h_j : t\}. \tag{2.2}$$

Define $t_j = t_j'$ *except* in each of the following two cases, in which we define $t_j = t_{j-1}$:

$$u \overset{\uparrow}{=} h_{j-1} : t_{j-1}, \qquad u \overset{\downarrow}{=} h_j : t_{j-1}, \qquad t_{j-2} < t_{j-1}; \tag{2.3a}$$

or

$$u = h_{j-1} : t_{j-1}, \qquad u \downarrow h_j : t_{j-1}, \qquad t_{j-2} = t_{j-1}. \tag{2.3b}$$

(Note that (2.3a) occurs in Fig. 2a, and (2.3b) occurs in Fig. 2b.)

The sequence $t_{-1}, t_0, t_1, \ldots$ is defined to be the *lower oscillation sequence* of u relative to $[h_0, h_1]$. The *lower oscillation* of u relative to $[h_0, h_1]$ is defined to be $\underline{O}(u) = \sup\{i \mid t_i < +\infty\}$. The *upper oscillation sequence* of u and the *upper oscillation* $\bar{O}(u)$ of u relative to $[h_0, h_1]$ are defined as above with h_j replaced by h_{j+1} everywhere in the definition of t_i, t_i' ($i = -1, 0, 1, \ldots$).

(2.4) *Note.* If $u \in]]h_0, h_1[[$ then $-u \in]]-h_1, -h_0[[$ and the lower oscillation sequence for $-u$ relative to $[-h_1, -h_0]$ is the upper oscillation sequence for u relative to $[h_0, h_1]$. Hence, it is sufficient to study lower oscillation sequences.

The proof of the next lemma follows by an elementary calculation.

(2.5) Lemma. *Let R be a bounded subset of $\mathbb{R}$ and let X, Y be sequentially compact subsets of $\mathscr{F}(R)$. Then*

$$S = \{t \in \operatorname{cl} R \mid \exists x \in X, y \in Y : x \succ y : t\}$$

is compact.

Since the remaining results in this section are concerned with oscillation sequences, which were defined only for bounded T, for the remainder of this section T is assumed to denote a bounded subset of $\mathbb{R}$.

The next two lemmas describe some basic features of oscillation sequences.

(2.6) Lemma. *Let $h_0, h_1 \in \mathscr{F}(T)$ be such that $]]h_0, h_1[[= \emptyset$. For any $u \in [h_0, h_1]$ let $t_{-1}, t_0, \ldots$ be the lower oscillation sequence of u relative to $[h_0, h_1]$. Then for $j = 0, 1, \ldots$*

$$t_{j-1} \leq t_j, \tag{2.6.1}$$

$$t_j < +\infty \Rightarrow u \succ h_j : t_j, \tag{2.6.2}$$

$$t \in]t_{j-1}, t_j[\Rightarrow u \not\cong h_j : t, \tag{2.6.3}$$

$$t_j < +\infty, t_{j-1} < t_j \Rightarrow u \doteq h_j : t_j. \tag{2.6.4}$$

Proof. In what follows, t_j' is as defined in (2.2).

(1) Since $t'_{-1} = t_{-1} = -\infty$ it is clear from the definition of t_j' that for $j = 0, 1, \ldots$

$$t_{j-1} \leq t_j'.$$

On the other hand, from the definition of t_j, either $t_j = t_{j-1}$ or $t_j = t_j'$. Hence

$$t_{j-1} \leq t_j \leq t_j'. \tag{a}$$

(2) As in (1), $t_j = t_{j-1}$ or $t_j = t_j'$. If $t_j = t_j'$ then from the definition of t_j, either (2.3a) or (2.3b) holds, so $u \doteq h_j : t_j$. If $t_j = t_j'$ we use (2.5) to show $u \cong h_j : t_j$. In fact, in this lemma, set

$$R =]t_{j-1}, +\infty[\cap T, \qquad X = \{u|_R\}, \qquad Y = \{h_j|_R\}.$$

Then by definition of t_j' it follows that

$$u \succ h_j : t_j' \quad t.$$

(3) Suppose $u \cong h_j : t$ for some $t \in]t_{j-1}, t_j[$; then from the definition of t_j' it follows that

$$t_{j-1} \leqslant t_j' \leqslant t < t_j ,$$

which contradicts (a). Hence

$$t \in]t_{j-1}, t_j[\Rightarrow u \not\cong h_j : t, \qquad j = 0, 1, \dots .$$

(4) As in the proof of (2), since $t_j < +\infty$, if $t_j \neq t_j'$ then $u \overset{\downarrow}{=} h_j : t_j$ from the definition of t_j. Now if

$$t_j = t_{j-1} = t_j'$$

then again as in (2), using Lemma 2.5 it follows that

$$u \downarrow h_j : t_j .$$

In the next lemma we prove some needed facts about the lower oscillation sequence for the case when u is assumed not to "touch" both h_0 and h_1 at the same point t, from the same side (which is the case in what follows).

(2.7) LEMMA. *Let $h_0, h_1 \in \mathscr{F}(T)$ be such that $]]h_0, h_1[[\neq \varnothing$. For any $u \in [h_0, h_1]$, let $t_{-1}, t_0, \dots$ be the lower oscillation sequence of u relative to $[h_0, h_1]$. If for every $t \in \mathrm{cl}\, T$*

$$\begin{array}{c} \text{NEITHER } (u \uparrow h_0 : t \text{ and } u \uparrow h_1 : t) \\ \text{NOR } (u \downarrow h_0 : t \text{ and } u \downarrow h_1 : t) \end{array} \qquad (*)$$

then for $j = 0, 1, \dots$

$$u \downarrow h_j : t_{j-1} \Rightarrow t_{j-1} = t_j ; \qquad (2.7.1)$$

$$t_j < +\infty,\ t_{j-1} = t_j \Rightarrow u \overset{\uparrow}{=} h_{j-1} : t_j \quad \text{and} \quad u \overset{\downarrow}{=} h_j : t_j ; \qquad (2.7.2)$$

$$t_j < +\infty,\ t_{j-1} = t_j = t_{j+1} \Rightarrow u \uparrow\downarrow h_{j+1} : t_j \quad \text{and} \quad u = h_j : t_j ; \qquad (2.7.3)$$

$$t_j < +\infty \Rightarrow t_j < t_{j+3} . \qquad (2.7.4)$$

Proof. (1) Assuming $u \downarrow h_j : t_{j-1}$ and $(*)$, it follows that $u \updownarrow h_{j-1} : t_{j-1}$. However, from (2.6.2) $u \cong h_{j-1} : t_{j-1}$. Hence $u \overset{\uparrow}{=} h_{j-1} : t_{j-1}$. Therefore, when $t_{j-2} < t_{j-1}$, (2.3) is satisfied so from the definition of t_j, $t_j = t_{j-1}$. When $t_{j-2} = t_{j-1}$, then from (2.6.4), $u \overset{\downarrow}{=} h_{j-1} : t_{j-1}$. But we just showed that $u \updownarrow h_{j-1} : t_{j-1}$, so $u = h_{j-1} : t_{j-1}$ and thus (2.4) is satisfied so, also from the definition of t_j, $t_j = t_{j-1}$.

(2) From (2.6.4) it follows that $u \overset{\downarrow}{=} h_j : t_j$. If $t_j \neq t_j'$ (t_j' as in (2.2)) it follows from the definition of t_j that either (2.3a) or (2.3b) holds, so $u \overset{\uparrow}{=} h_{j-1} : t_{j-1} = t_j$. If $t_j = t_{j-1} = t_j'$, then as in the proof of (2.6.4) it follows that $u \downarrow h_j : t_{j-1} = t_j$. By $(*)$, $u \updownarrow h_{j-1} : t_{j-1}$. But $u \cong h_{j-1} : t_{j-1}$ by (2.6.2) so again we obtain $u \overset{\downarrow}{=} h_{j-1} : t_{j-1} = t_j$.

(3) Using (2.7.2) twice, get ($u \sim h_{j-1} : t_j$ and $u \sim h_j : t_j$) and ($u \sim h_j : t_j$ and $u \sim h_{j-1} : t_j$). If $t_{j+1} = t'_{j+1}$ then it follows from definition of t_{j+1} that $u \downarrow h_{j+1} : t_j$ whence it follows from $(*)$ that $u \overset{+}{\downarrow} h_j : t_j$, so $u \sim h_j : t_j$ and $u \downarrow\uparrow h_{j+1} : t_j$. If $t_{j+1} = t_j = t'_{j+1}$ then as in the proof of (2.6.4), $u \downarrow h_{j+1} : t_j$, so again as before, $u \sim h_j : t_j$ and $u \downarrow\uparrow h_{j+1} : t_j$.

(4) Follows trivially from (3) using $(*)$.

The two theorems which follow show that if $\underline{O}(u) = n$, although several t_i's may coincide, the function u really does "oscillate" n times between h_0 and h_1.

(2.8) THEOREM. *Let $h_0, h_1 \in \mathscr{F}(T)$ be such that $]]h_0, h_1[[= \emptyset$. For any $u \in [h_0, h_1]$, every $\epsilon > 0$ and any integer n such that $0 \leqslant n \leqslant \underline{O}(u)$, there exist $s_j \in T$, $j = 0, 1, \ldots, n$ satisfying*

$$s_{j-1} < s_j, \qquad j = 1, 2, \ldots, n; \tag{2.8.1}$$

$$0 \leqslant (-1)^j (u(s_j) - h_j(s_j)) < \epsilon, \qquad j = 0, 1, \ldots, n. \tag{2.8.2}$$

Proof. If $(*)$ of Lemma 2.7 does not hold, then for some $t \in \text{cl } T$ there are either two strictly increasing or two strictly decreasing sequences (a_i), (b_i), $i = 1, 2, \ldots$, such that

$$a_i \to t, \qquad b_i \to t,$$

and

$$|u(a_i) - h_0(a_i)| < \epsilon,$$
$$|u(b_i) - h_1(b_i)| < \epsilon.$$

The required s_j's can be selected as elements chosen alternatingly from the two sequences (a_i), (b_i).

Now assume that $(*)$ of Lemma 2.7 holds for all $t \in \text{cl } T$. Let n be chosen so that $0 \leqslant n \leqslant \underline{O}(u)$ and let $-\infty, t_0, \ldots, t_n, \ldots$ be the lower oscillation sequence of u. Since $n \leqslant \underline{O}(u)$, $t_j < +\infty$ for $j = 0, \ldots, n$. If $n = 0$, there is an $s_0 \in T$ such that $|u(s_0) - h_0(s_0)| < \epsilon$ (since $u \simeq h_0 : t_0$). Hence, assume now that $n > 0$. Define

$$\delta = \tfrac{1}{2} \min(t_j - t_{j-1}), \qquad j = 1, \ldots, n, \qquad t_{j-1} < t_j.$$

We next choose $s_j \in T$ for $0 \leqslant j \leqslant n$. By (2.7.4), for $j \leqslant n$, $t_j < t_{j+3}$ and hence there are four germane possibilities to consider:

(a) $t_{j-1} < t_j < t_{j+1}$, in which case since $u \sim h_j : t_j$ we choose $s_j \in T$ such that $|t_j - s_j| < \delta$ and (2.8.2) is satisfied:

(b) $t_{j-1} = t_j < t_{j+1}$, in which case by (2.7.2) $u \overset{\downarrow}{=} h_j : t_j$ so we find $s_j \in T$ such that

$$t_j \leqslant s_j < t_j + \delta,$$

$$s_j = t_j \quad \text{iff} \quad u = h_j : t_j$$

and (2.8.2) is satisfied;

(c) $t_{j-1} < t_j = t_{j+1}$, in which case by (2.7.2) $u \overset{\uparrow}{=} h_j : t_{j+1} = t_j$ so we find $s_j \in T$ such that

$$t_j - \delta < s_j \leqslant t_j,$$

$$s_j = t_j \quad \text{iff} \quad u = h_j : t_j$$

and (2.8.2) is satisfied;

(d) $t_{j-1} = t_j = t_{j+1}$, in which case by (2.7.3) $u = h_j : t_j$ so $t_j \in T$ and we set $s_j = t_j$, whence (2.8.2) is satisfied.

It remains to verify (2.8.1) for the above four cases:

(a) Since in all cases $|t_i - s_i| < \delta$ for $i = 0,..., n$, it follows that $s_{j-1} < t_{j-1} + \delta \leqslant t_j - \delta < s_j$;

(b) Since $t_{j-1} = t_j$, case (c) or (d) applies in the definition of t_{j-1} and in each of these cases $s_{j-1} \leqslant t_j$. By the definition of s_j in case (b), $t_j \leqslant s_j$ so $s_{j-1} \leqslant t_j \leqslant s_j$. But $s_{j-1} = t_j$ only if $u = h_{j-1} : t_j$ whence $u \neq h_j : t_j$ so $t_j < s_j$ and thus in either case $s_{j-1} < s_j$;

(c) As in (a) we have $s_{j-1} < t_{j-1} + \delta \leqslant t_j - \delta < s_j$;

(d) Since $u = h_j : t_j$, $u \neq h_{j-1} : t_j = t_{j-1}$. But in the definition of s_{j-1} case (c) must apply since $t_{j-2} < t_{j+1} = t_{j-1}$ by (2.7.4) so $s_{j-1} < t_j$. Hence $s_{j-1} < t_j = s_j$.

3. Tchebycheff Spaces

For notation, see [2].

Note that if $U \subset \mathscr{F}(T)$ is a T-space, then so is $U|_S$, whenever $S \subset T$ satisfies card $S \geqslant \dim U$ (where $U|_S$ is the set of restrictions of the elements of U to S).

The next result shows that if u and some $v \in]]h_0, h_1[[$ are both in a T-space, then condition (*) of Lemma 2.7 holds and hence the consequences (2.7.1) to (2.7.4) apply.

(3.1) LEMMA. *Let $T \subset \mathbb{R}$ and let $h_0, h_1 \in \mathscr{F}(T)$. If for some $v \in]]h_0, h_1[[$, an element $u \in [h_0, h_1]$ satisfies $S^-(u - v) < +\infty$, then for every $t \in \text{cl}\, T$*

$$\begin{aligned} &\text{NEITHER } (u \uparrow h_0 : t \text{ and } u \uparrow h_1 : t) \\ &\quad \text{NOR } (u \downarrow h_0 : t \text{ and } u \downarrow h_1 : t). \end{aligned} \tag{*}$$

Proof. As in the first paragraph of the proof of Theorem 2.8.

In the following theorem we give an upper bound to $\underline{O}(u)$ and $\bar{O}(u)$ relative to $[h_0, h_1]$ when u is an element of a T-space U and $]]h_0, h_1[[\cap U \neq \emptyset$, namely if the degree of U is k, then

$$\underline{O}(u) \leqslant k, \qquad \bar{O}(u) \leqslant k. \tag{3.2}$$

(3.3) THEOREM. *Let T be a bounded subset of $\mathbb{R}$ and let $h_0, h_1 \in \mathscr{F}(T)$. For any $u \in [h_0, h_1]$ and $v \in]]h_0, h_1[[$*

$$\max\{\underline{O}(u), \bar{O}(u)\} \leqslant S^-(u - v).$$

Proof. With $u \in [h_0, h_1]$ and $v \in]]h_0, h_1[[$ we have $u - v \in [h_0 - v, h_1 - v]$ and $0 \in]]h_0 - v, h_1 - v[[$. Since $\underline{O}(u)(\bar{O}(u))$ relative to $[h_0, h_1]$ is equal to $\underline{O}(u - v)(\bar{O}(u - v))$ relative to $[h_0 - v, h_1 - v]$, it suffices to prove the theorem for $v = 0$.

From Theorem 2.8 it follows that for any $\epsilon > 0$ and any integer n such that $0 \leqslant n \leqslant \underline{O}(u)$, there exist $s_0, \ldots, s_n$ satisfying (2.8.1) and (2.8.2). Use $\epsilon = \frac{1}{2} \inf_{t \in T} \min\{|h_0(t)|, |h_1(t)|\}$ which is greater than zero since $0 \in]]h_0, h_1[[$. Then, from (2.8.2), $(-1)^i u(s_i) < \epsilon + (-1)^i h_i(s_i)$. From the definition of ϵ and the fact that $(-1)^i h_i(s_i) < 0$, it follows that $(-1)^i u(s_i) < 0$ for $i = 0, \ldots, n$, whence $n \leqslant S^-(u)$. It follows that $\underline{O}(u) \leqslant S^-(u)$.

An analogous proof shows that $\bar{O}(u) \leqslant S^-(u)$, and this completes the proof.

4. THE COMPACTNESS OF $[h_0, h_1] \cap U$

Let $T \subset \mathbb{R}$ and suppose U is a finite-dimensional linear subspace of $\mathscr{F}(T)$ with basis $u_0, \ldots, u_k$. Then the vector space isomorphism $U \simeq \mathbb{R}^{k+1}$ defined by

$$\sum_{i=0}^{k} c_i u_i \to (c_0, \ldots, c_k)$$

induces the l_2 norm on U.

On the other hand, if $U \subset \mathscr{B}(T)$ then U is already normed by the sup norm. It is well known that the topologies defined by any two norms on a finite-dimensional vector space are the same.

The proofs of the following results are left to the reader.

(4.1) THEOREM. *Let $T \subset \mathbb{R}$, let $h_0, h_1 \in \mathscr{F}(T)$, and suppose U is a $(k+1)$-dimensional subspace of $\mathscr{F}(T)$. Then $[h_0, h_1] \cap U$ is l_2-compact in U.*

(4.2) COROLLARY. *Let $T \subset \mathbb{R}$, let $h_0, h_1 \in \mathscr{F}(T)$ and suppose U is a $(k+1)$-dimensional subspace of $\mathscr{B}(T)$. Then $[h_0, h_1] \cap U$ is compact (in the induced sup-norm topology) in U.*

(4.3) LEMMA. *Let $X \subset \mathscr{F}(T)$ and $f \in \mathscr{F}(T)$. Then for each $t \in \operatorname{cl} T$, if X is sequentially compact in $\mathscr{F}(T)$ so are each of the sets*

$$Y_- = \{u \in X \mid u \uparrow f : t\}, \quad Y = \{u \in X \mid u = f : t\}, \quad Y_+ = \{u \in X \mid u \downarrow f : t\}.$$

(4.4) COROLLARY. *Let $U \subset \mathscr{B}(T)$ be a finite-dimensional linear subspace, let $X \subset U$, and let $f \in \mathscr{F}(T)$. Then for each $t \in \operatorname{cl} T$, if X is compact so are each of the sets*

$$\{u \in X \mid u \cong f : t\}, \qquad \{u \in X \mid u \overset{\downarrow}{=} f : t\}, \qquad \{u \in X \mid u \downarrow f : t\}.$$

5. THE OSCILLATION THEOREM

In this section we prove the Oscillation Theorem, which shows the existence of the function $\underline{u}$, $\bar{u}$ described in the introduction. As the properties obtained in Lemmas 2.6, 2.7, and 3.1 are used frequently, for easy reference we list them below.

We are concerned exclusively with the case in which the conditions of Lemma 3.1 are satisfied, so we have for every $t \in \operatorname{cl} T$ and $u \in [h_0, h_1] \cap U$:

$$\begin{aligned} &\text{NEITHER } (u \uparrow h_0 : t \text{ and } u \uparrow h_1 : t) \\ &\quad\text{NOR } (u \downarrow h_0 : t \text{ and } u \downarrow h_1 : t). \end{aligned} \tag{$*$}$$

Therefore also the conclusions of Lemma 2.7 obtain. We next summarize the needed consequences of Lemmas 2.6 and 2.7. Let $t_{-1}, t_0, \ldots$ be the lower oscillation sequence of u relative to $[h_0, h_1]$. Then for $j = 0, 1, \ldots$

$$t_{j-1} \leqslant t_j, \tag{a}$$

$$t_j < +\infty \Rightarrow u \cong h_j : t_j, \tag{b}$$

$$t \in \,]t_{j-1}, t_j[\, \Rightarrow u \not\cong h_j : t, \tag{c}$$

$$t_j < +\infty,\ t_{j-1} = t_j \Rightarrow u \overset{\downarrow}{=} h_j : t_j \quad \text{and} \quad u \overset{\uparrow}{=} h_{j-1} : t_j, \tag{d}$$

$$u \downarrow h_j : t_{j-1} \Rightarrow t_{j-1} = t_j, \tag{e}$$

$$t_j < +\infty,\ t_{j-1} = t_j = t_{j+1} \Rightarrow u \uparrow\downarrow h_{j+1} : t_j \quad \text{and} \quad u = h_j : t_j, \tag{f}$$

$$t_j < +\infty \Rightarrow t_j < t_{j+3}. \tag{g}$$

(5.1) THEOREM. *Let T be a bounded subset of $\mathbb{R}$, and suppose $U \subset \mathscr{B}(T)$ is a T-space of degree k. Given any two real-valued functions h_0, h_1 on T such that $]]h_0, h_1[[\cap U \neq \varnothing$, there exist $\underline{u}, \bar{u} \in U$ such that the lower oscillation of $\underline{u}$ and the upper oscillation of $\bar{u}$, relative to $[h_0, h_1]$, are both equal to k.*

Proof. In view of Note 2.4 it is sufficient to prove the theorem for $\underline{u}$. Since U is a T-space of degree k on T, card $T > k$. Also if $p \in]]h_0, h_1[[\cap U$ then $0 \in]]h_0 - p, h_1 - p[[\cap U$ and if $Q(\underline{u}) = k$ relative to $[h_0 - p, h_1 - p]$ then $Q(\underline{u} + p) = k$ relative to $[h_0, h_1]$. Hence it is sufficient to assume that $0 \in]]h_0, h_1[[$. For any $u \in [h_0, h_1] \cap U$, let $t_{-1}(u), t_0(u), t_1(u),\ldots$ denote the lower oscillation sequence of u relative to $[h_0, h_1]$. We then define r_i, U_i, $i = -1, 0, 1,\ldots$, as follows. Let $r_{-1} = -\infty$ and $U_{-1} = [h_0, h_1] \cap U$. For $i = 0, 1, 2,\ldots$ define

$$r_i = \inf\{t_i(u) : u \in U_{i-1}\},$$

$$U_i = \{u \in U_{i-1} : t_i(u) = r_i\}.$$

Notice that

$$u \in U_i \Rightarrow t_j(u) = r_j, \quad \text{for } j \leq i. \tag{1}$$

Therefore r_j for $j \leq i$ satisfy all the properties (a) to (g) of lower oscillation sequences listed above.

Also, from (1) and (3.2) it follows that

$$u \in U_i \Rightarrow i \leq Q(u) \leq k \tag{2}$$

so in particular, $r_i = +\infty$ and $U_i = \varnothing$ for $i > k$.

We next show by induction on j that for $j = -1, 0,\ldots$

$$r_j < +\infty \Rightarrow U_j \neq \varnothing \quad \text{and} \quad U_j \text{ is compact.} \tag{3}$$

This is clearly true for $j = -1$ because by assumptions of the theorem, $U_{-1} = [h_0, h_1] \cap U \neq \varnothing$ and from Corollary 4.2 U_{-1} is compact. Suppose (3) is true for $j < i$ and that $r_i < +\infty$. Then $r_{i-1} < +\infty$, since $r_{i-1} \leq r_i$ by (a). Hence by the induction assumption $U_{i-1} \neq \varnothing$ and U_{i-1} is compact. In this case if $U_i = \varnothing$ then for all $u \in U_{i-1}$

$$t_{i-1}(u) = r_{i-1} \leq r_i < t_i(u). \tag{4}$$

Therefore, when $U_i = \varnothing$, from the definition of $t_i(u)$, (2.2) applies and for each $u \in U_{i-1}$ we have

$$t_i(u) = \inf\{t > r_{i-1} : u \geqq h_i : t\}. \tag{5}$$

Hence from the definition of r_i

$$r_i = \inf_{u \in U_{i-1}} \{t > r_{i-1} : u \geqq h_i : t\}.$$

We now apply Lemma 2.5 by setting $R =]r_{i-1}, +\infty[\cap T$, $X = U_{i-1}|_R$ and $Y = \{h_i|_R\}$. Since X is a continuous image of the compact U_{i-1}, it is compact. Hence Lemma 2.5 implies that there is some $u \in U_{i-1}$ such that $u|_R \cong h_i|_R : r_i$, whence $u \cong h_i : r_i$ and since $r_{i-1} \notin R$, $u \downarrow h_i : r_i$ if $r_i = r_{i-1}$. If $r_i > r_{i-1}$ then from (5), $t_i(u) \leqslant r_i$. If $r_i = r_{i-1}$ then from (e) it follows that $t_i(u) = t_{i-1}(u) = r_{i-1} = r_i$. Therefore in either case $t_i(u) \leqslant r_i$, which contradicts (4). Hence $U_i \neq \varnothing$.

We now show that U_i is compact. If $u \in U_i$ then $r_j = t_j(u)$ for $j \leqslant i$. It follows from (g) that

$$r_{i-3} < r_i .$$

Hence, by (b), (d), and (f), U_i can be expressed in one of three ways:

$$\begin{aligned} &\text{if} \quad r_i > r_{i-1} && \text{then} \quad U_i = \{u \in U_{i-1} \mid u \cong h_i : r_i\}, \\ &\text{if} \quad r_i = r_{i-1} > r_{i-2} && \text{then} \quad U_i = \{u \in U_{i-1} \mid u \stackrel{\downarrow}{=} h_i : r_i\}, \\ &\text{if} \quad r_i = r_{i-1} = r_{i-2} && \text{then} \quad U_i = \{u \in U_{i-1} \mid u \downarrow h_i : r_i\}. \end{aligned}$$

In each of these cases it follows from Corollary 4.4 that U_i is compact. Hence if $r_i < +\infty$ then $U_i \neq \varnothing$ and U_i is compact, which proves (3).

Let $a = \inf\{t \in \mathbb{R} \mid \operatorname{card}]-\infty, t] \cap T > k\}$. Since card $T > k$ it follows that $a < +\infty$. Hence, for some $\nu \leqslant k$,

$$r_\nu < a \leqslant r_{\nu+1} .$$

Now we define a linear form δ on U as described in the Introdutcion. We show that the element $\underline{u}$, in U_ν, $U_{\nu+1}$, or $U_{\nu+2}$, depending respectively upon whether $a < r_{\nu+1}$, $a = r_{\nu+1} < r_{\nu+2}$ or $a = r_{\nu+1} = r_{\nu+2}$, which minimizes δ there, satisfies $Q(\underline{u}) = k$.

Since $\operatorname{card}]-\infty, a] \cap T < +\infty$ it follows that $u \updownarrow h_i : a$ for any i, and any $u \in [h_0, h_1] \cap U$. Hence, if $a = r_{\nu+1}$ then $r_{\nu+3} > r_{\nu+1}$ since $a = r_{\nu+1} = r_{\nu+2} = r_{\nu+3} \Rightarrow u \uparrow h_{\nu+3} : a$ by (f). We define λ in each of the three cases: $a < r_{\nu+1}$, $a = r_{\nu+1} < r_{\nu+2}$, $a = r_{\nu+1} = r_{\nu+2} < r_{\nu+3}$ as $\lambda = \nu, \nu + 1, \nu + 2$, respectively. Then

$$r_\lambda \leqslant a < r_{\lambda+1} ,$$

so from the definition of a,

$$\operatorname{card} \bigcup_{i=0}^{\lambda+1} (]r_{i-1}, r_i[\cap T) \geqslant k - \lambda.$$

Let ω_{ij}, $i = 0, 1,..., \lambda + 1$, be $k - \lambda$ points such that

$$\omega_{ij} \in]r_{i-1}, r_i[\cap T \qquad \text{and} \qquad \omega_{ij} < \omega_{ij+1} .$$

Let δ be the linear form on U defined by

$$\delta(u) = \sum_{j=0}^{\lambda+1} (-1)^j \sum_i u(\omega_{ji}).$$

Since $r_\lambda \leqslant a < +\infty$, U_λ is compact from (3). Thus, there is a $\underline{u} \in U$ which minimizes δ over U_λ. We show that $\underline{Q}(\underline{u})$ must equal k. Suppose $\underline{Q}(\underline{u}) < k$. Let $t_{-1}, t_0, \ldots$ be the lower oscillation sequence of $\underline{u}$ relative to $\{h_0, h_1\}$. Since $\underline{u} \in U_\lambda$, from (2) it follows that $\lambda \leqslant \underline{Q}(\underline{u})$ and

$$t_j = r_j \qquad \text{for} \quad j \leqslant \lambda.$$

We show that if $\underline{Q}(\underline{u}) < k$ then we can find a $\phi \in U$ such that for some $\epsilon > 0$, $\underline{u} + \epsilon\phi \in U_\lambda$ and $\delta(\underline{u} + \epsilon\phi) < \delta(\underline{u})$, which contradicts the definition of $\underline{u}$. It follows, in view of (2), that $\underline{Q}(\underline{u}) = k$.

The choice of ϕ is dependent on the lower oscillation sequence of $\underline{u}$ and is defined in terms of its zeros z_j's. Specifically we define $z_j \in \mathrm{cl}\, T$ as follows for $j = 0, 1, \ldots, \underline{Q}(\underline{u})$.

If

$$\underline{u} \mathrel{\hat{\uparrow}} h_j : t_j \qquad \text{and} \qquad t_{j-1} < t_j \tag{6}$$

let $z_j' = \sup\{t \in]t_{j-1}, t_j[\mid \underline{u} \simeq h_{j+1} : t\}$. Then $z_j' < t_j$ because $z_j' = t_j$ implies from Lemma 2.5 that $\underline{u} \uparrow h_{j+1} : t_j$; however, from (*) $\underline{u} \mathrel{\hat{\uparrow}} h_j : t_j \Rightarrow \underline{u} \mathrel{\hat{\not\uparrow}} h_{j+1} : t_j$. Therefore in this case $]z_j', t_j[\cap T \neq \emptyset$ (recall $\underline{u} \mathrel{\hat{\uparrow}} h_j : t_j$) and

$$t \in]z_j', t_j[\Rightarrow \underline{u} \not\simeq h_{j+1} : t. \tag{7}$$

Let z_j be any element in $]z_j', t_j[\cap T$, unless $j = \lambda + 1$; we choose $z_{\lambda+1} \in]z'_{\lambda+1}, t_{\lambda+1}[\cap T$ to additionally satisfy

$$\max_i \{\omega_{\lambda+1\, i}\} < z_{\lambda+1}. \tag{8}$$

This is possible since $t_{\lambda+1} \geqslant r_{\lambda+1} > \omega_{\lambda+1\, i}$ for every i.

When (6) does not hold, then

$$u \mathrel{\hat{\not\uparrow}} h_j : t_j \qquad \text{or} \qquad t_{j-1} = t_j \tag{9}$$

in which case we define $z_j = t_j$. Define $z_{-1} = -\infty$ and $z_{\underline{Q}(\underline{u})+1} = +\infty$.

From the above it is clear that for $j = 0, 1, \ldots$

$$z_{j-1} \leqslant t_{j-1} < z_j \leqslant t_j. \tag{10}$$

Let $Z_j =]z_{j-1}, z_j[\cap T$ for $j = 0, 1, \ldots, \underline{Q}(\underline{u}) + 1$. We next show that if $t \in \mathrm{cl}\, Z_j$ then

$$u|_{Z_j} \not\simeq h_j|_{Z_j} : t \tag{11}$$

for $j = 0, 1, \ldots, \underline{Q}(\underline{u}) + 1$. Indeed, from (10)

$$\operatorname{cl} Z_j \subset]z_{j-1}, t_{j-1}[\cup]t_{j-1}, z_j[\\ \cup (]z_{j-1}, z_j[\cap \{t_{j-1}\}) \cup \{z_{j-1}\} \cup \{z_j\}.$$

If $t \in]z_{j-1}, t_{j-1}[$ then $z'_{j-1} < z_{j-1} < t < t_{j-1}$ so (11) follows from (7). If $t \in]t_{j-1}, z_j[$ then $t_{j-1} < t < z_j \leqslant t_j$ so (11) follows from (c). If $t = t_{j-1} \in]z_{j-1}, z_j[$ then $z_{j-1} < t_{j-1} < z_j$ and hence $\underline{u} \uparrow h_{j-1} : t_{j-1}$ (otherwise (9) applies and $t_{j-1} = z_{j-1}$). Therefore from $(*)$, $\underline{u} \updownarrow h_j : t_{j-1}$. On the other hand $t_{j-2} \leqslant z_{j-1} < t_{j-1}$ so if $\underline{u} \stackrel{\downarrow}{=} h_j : t_{j-1}$ then $t_j = t_{j-1}$ from the definition of t_j (2.3), whence $t_{j-1} = z_j = t_j$ from (10), a contradiction. Thus, if $t = t_{j-1} \in]z_{j-1}, z_j[$ then (11) holds. Continuing, if $t = z_{j-1}$ and $\underline{u}\,|_{Z_j} \cong h_j\,|_{Z_j} : z_{j-1}$ then $Z_j \neq \varnothing$ and $\underline{u} \downarrow h_j : z_{j-1}$, in which case $z_{j-1} = t_{j-1}$ (otherwise, $z'_{j-1} < z_{j-1} < t_{j-1}$ so $\underline{u} \not\cong h_j : z_{j-1}$ by (7)). Therefore $\underline{u} \downarrow h_j : t_{j-1}$, whence from (e), $z_{j-1} = t_{j-1} = t_j$. However, from (10), $z_{j-1} = z_j$ so $Z_j = \varnothing$, a contradiction. Hence (11) obtains for $t = z_{j-1}$. Finally, if $t = z_j$ and $\underline{u}\,|_{Z_j} \cong h_j\,|_{Z_j} : z_j$ then $Z_j \neq \varnothing$ and $\underline{u} \uparrow h_j : z_j$. If $z_j < t_j$ then (6) applies and hence $z_j \in]z_j', t_j[\subset]t_{j-1}, t_j[$, whence $\underline{u} \updownarrow h_j : z_j$ by (c). Hence $z_j = t_j$. Thus $\underline{u} \uparrow h_j : t_j$, in which case if $t_{j-1} < t_j$ then by (6), $z_j < t_j$ $(= z_j)$ so in fact $t_{j-1} = t_j$ and $\underline{u} \uparrow h_j : t_{j-1}$. Therefore, from $(*)$, $\underline{u} \updownarrow h_{j-1} : t_{j-1}$ and (9) applies. Hence $z_{j-1} = t_{j-1}$ $(= t_j = z_j)$ so again $Z_j = \varnothing$, a contradiction. Thus (11) holds for $t = z_j$. This completes the proof that $t \in \operatorname{cl} Z_j \Rightarrow$ (11).

We next use [3, Theorem 2.4] to find the desired ϕ as described previously. For this we define sequences $(z_{jn})_n \subset T$ for $0 \leqslant j \leqslant \underline{Q}(\underline{u})$ as follows. From (1), the definition of λ and by (a)

$$t_j = r_j \leqslant a \qquad \text{for} \quad j \leqslant \lambda.$$

Since $\operatorname{card}]-\infty, a] \cap T < +\infty$ it follows that

$$\underline{u} \not\sim h_j : t_j \qquad \text{for} \quad j < \lambda \quad \text{and} \quad \underline{u} \updownarrow h_\lambda : t_\lambda .$$

Thus (9) applies so

$$z_j = t_j \ (= r_j) \qquad \text{for} \quad j \leqslant \lambda.$$

Also, by (b) and (if $t_{\lambda-1} = t_\lambda$) by (d), $u = h_j : t_j$ $(j < \lambda)$ so

$$z_j = t_j \in T \qquad \text{for} \quad j < \lambda, \\ \underline{u} \updownarrow h_\lambda : t_\lambda \Rightarrow z_\lambda = t_\lambda \in T.$$

For $j = 0, 1, \ldots, \lambda - 1$ and for all $n \in \mathbb{N}$ define $z_{jn} = z_j$. Now, notice that from previous steps we have

$$z_\lambda = t_\lambda = r_\lambda \leqslant a < r_{\lambda+1} \leqslant t_{\lambda+1}$$

so by the definition of $z_{\lambda+1}$,

$$z_{\lambda+1} \in]t_\lambda, t_{\lambda+1}],$$

whence $t_\lambda < z_{\lambda+1}$; that is,

$$z_\lambda < z_{\lambda+1}.$$

For all $n \in \mathbb{N}$, if $\underline{u} \not\downarrow h_\lambda : r_\lambda$ define $z_{\lambda n} = z_\lambda$ and if $\underline{u} \downarrow h_\lambda : r_\lambda$ let $(z_{\lambda n})_n \subset T$ be a sequence such that $z_\lambda < z_{\lambda n} < z_{\lambda+1}$, $\lim_n z_{\lambda n} = z_\lambda$ and $\lim_n (\underline{u}(z_{\lambda n}) - h_\lambda(z_{\lambda n})) = 0$. Finally, when $\lambda < j \leqslant Q(\underline{u})$ if $z_j \in T$ define $z_{jn} = z_j$ for all $n \in \mathbb{N}$ and if $z_j \notin T$ let $(z_{jn})_n$ be any sequence such that $z_{jn} \in T$, $\lim_n z_{jn} = z_j$ and (defining inductively) such that $z_{j-1n} < z_{jn}$. Since it was assumed that $Q(\underline{u}) < k$, the conditions of [3, Theorem 2.4] are all satisfied with respect to z_j and $(z_{jn})_n$, $j = 0, 1, \ldots, Q(\underline{u})$.

Hence, there is a $\phi \in U$ such that

$$\|\phi\| = 1, \tag{12}$$

$$\phi(z_j) = 0 \qquad \text{if} \quad z_j \in T, \qquad j = 0, 1, \ldots, Q(\underline{u}), \tag{13}$$

$$\underline{u} \downarrow h_\lambda : r_\lambda \Rightarrow \underline{u} + \epsilon\phi \downarrow h_\lambda : r_\lambda \qquad \forall \epsilon > 0 \tag{14}$$

$$t \in Z_j \Rightarrow (-1)^j \phi(t) > 0, \qquad j = 0, 1, \ldots, Q(\underline{u}) - 1. \tag{15}$$

We now argue by contradiction that with this choice of ϕ there is an $\epsilon > 0$ such that $\underline{u} + \epsilon\phi \in [h_0, h_1]$. Indeed, suppose that for every $\epsilon > 0$, $\underline{u} + \epsilon\phi \notin [h_0, h_1]$. Then there is a sequence $(x_n) \subset T$ such that with $i = 0$, or $i = 1$: for every $n \in \mathbb{N}$, $(-1)^i(\underline{u} + n^{-1}\phi)(x_n) < (-1)^i h_i(x_n)$.

Since $\underline{u} \in [h_0, h_1]$, $(-1)^i(h_i - \underline{u})(x) \leqslant 0$ for all $x \in T$. Therefore

$$(-1)^i n^{-1}\phi(x_n) < (-1)^i(h_i(x_n) - \underline{u}(x_n)) \leqslant 0.$$

Furthermore, from (12), $\lim_n n^{-1}\phi(x_n) = 0$. Therefore $\lim_n(h_i(x_n) - \underline{u}(x_n)) = 0$. Recapitulating, if for every $\epsilon > 0$, $\underline{u} + \epsilon\phi \notin [h_0, h_1]$, then there exists a sequence $(x_n) \subset T$ such that

$$(-1)^i \phi(x_n) < 0 \qquad \text{and} \qquad \lim_n (h_i(x_n) - \underline{u}(x_n)) = 0. \tag{16}$$

By taking a subsequence of (x_n) if necessary, it may be assumed that (x_n) is either strictly monotonic or else constant. Since $T \subset \bigcup \mathrm{cl}\, Z_j$ ($j = 0, 1, \ldots, Q(\underline{u}) + 1$), one of the two following cases must hold:

$$x_n = z_j \qquad \text{for some } j \text{ and all } n; \tag{I}$$

OR

$$x_n \in Z_j \qquad \text{for some } j \text{ and all sufficiently large } n. \tag{II}$$

We now show that each of these two cases leads to a contradiction of (16).

Case (I). Since $(x_n) \subset T$, it follows that $z_j \in T$, whence from (13), $\phi(z_j) = 0$; i.e., $\phi(x_n) = 0$, which shows that (16) cannot hold.

Case (II). If (16) does hold then for $x = \lim_n x_n$, $x \in \text{cl}\, Z_j$ and $\underline{u}|_{Z_j} \cong h_i|_{Z_j} : x$ which from (11) implies that $i \not\equiv j \pmod 2$. On the other hand, from (15), if $x_n \in Z_j$ and $(-1)^i \phi(x_n) < 0$ then $i \equiv j \pmod 2$. This contradiction proves that also in this case (16) cannot hold.

Therefore, for some $\epsilon > 0$, since $\underline{u}, \phi \in U$,

$$\underline{u} + \epsilon\phi \in [h_0, h_1] \cap U = U_{-1}.$$

We next show that $\underline{u} + \epsilon\phi \in U_\lambda$. Indeed, since for $j < \lambda$, $z_j = r_j \in T$, by (13), $\phi(r_j) = 0$, whence

$$\underline{u} + \epsilon\phi \in U_{\lambda-1}.$$

If $\underline{u} \updownarrow h_\lambda : r_\lambda$ then also $z_\lambda = r_\lambda \in T$ and $\phi(r_\lambda) = 0$, so $\underline{u} + \epsilon\phi \in U_\lambda$. If $\underline{u} \downarrow h_\lambda : r_\lambda$ then by (14), $\underline{u} + \epsilon\phi \downarrow h_\lambda : r_\lambda$ and so in either case

$$\underline{u} + \epsilon\phi \in U_\lambda. \tag{17}$$

Finally we are able to show that the definition of $\underline{u}$ as the element of U_λ which minimizes δ, is contradicted by (17), which in turn was derived from the assumption that $\underline{O}(\underline{u}) < k$.

Indeed, since δ is linear, $\delta(\underline{u} + \epsilon\phi) = \delta(\underline{u}) + \epsilon\delta(\phi)$. It remains only to show that $\delta(\phi) < 0$, as then $\delta(\underline{u} + \epsilon\phi) < \delta(\underline{u})$, which is a contradiction. But $\delta(\phi) = \sum_{i=0}^{\lambda+1} (-1)^i \sum_j \phi(\omega_{ij})$, where $\omega_{ij} \in \,]r_{i-1}, r_i[\, \cap T = Z_i$ for $i = 0, 1,\dots, \lambda$ and $\omega_{\lambda+1j} \in \,]r_\lambda, z_{\lambda+1}[\, \cap T = Z_{\lambda+1}$ by (8). Also, by (2), $\lambda \leqslant \underline{O}(\underline{u})$ and thus by (15), $\delta(\phi) \leqslant 0$. However, by (13), ϕ has at least $\underline{O}(\underline{u}) + 1$ distinct zeros, namely z_j for $j = 0, 1,\dots, \underline{O}(\underline{u})$ and these are all distinct from the ω_{ij}'s of which there are $k - \lambda \geqslant k - \underline{O}(\underline{u})$. Since $\phi \in U$, $Z(\phi) \leqslant k$ and thus ϕ must be nonzero at no fewer than $k - \underline{O}(\underline{u}) - (k - (\underline{O}(\underline{u}) + 1)) = 1$ point ω_{ij}, whence $\delta(\phi) < 0$.

(5.2) COROLLARY. *If h is any positive element of a T-space U of degree k such that $]]0, h[[\neq \varnothing$ then*

$$h = \underline{u} + \bar{u},$$

where $\underline{u}, \bar{u}$ are nonnegative elements of $U \cap [0, h]$ such that $\underline{O}(\underline{u}) = \bar{O}(\bar{u}) = k$ relative to $[0, h]$.

Proof. Let $\underline{u}$ be the element of $U \cap [0, h]$ such that $\underline{O}(\underline{u}) = k$, given by Theorem 5.1. Then from Note 2.4, $\bar{O}(-\underline{u})$ relative to $[-h, 0]$ is k, whence $\bar{O}(h - \underline{u})$ relative to $[0, h]$ is k. Set $\bar{u}$ to $h - \underline{u}$ and the proof is complete.

(5.3) COROLLARY. *Let $U \subset \mathscr{B}(T)$ be any T-space of degree k on a bounded set $T \subset \mathbb{R}$. Given any $h_0, h_1 \in \mathscr{F}(T)$ with $]]h_0, h_1[[\cap U \neq \varnothing$, there exists a $\underline{u} \in [h_0, h_1] \cap U$ such that*

$$v \in \,]]h_0, h_1[[\;\Rightarrow S^-(\underline{u} - v) \geqslant k.$$

Proof. Follows from Theorems 5.1 and 2.8.

(5.4) *Notes.* (1) The definitions here of $\underline{u}$ and $\bar{u}$ differ from those of Karlin and Studden [6] to the extent that when k is odd, what we define to be $\underline{u}$, they define to be $\bar{u}$ and conversely. The Karlin and Studden definitions presumably derive from the desire to have $\underline{u}$ and $\bar{u}$ correspond in kind to $\underline{\sigma}$ and $\bar{\sigma}$, two mass distribution functions which determine, respectively, lower and upper bounds to a classical problem in the theory of moment spaces. In order to avoid unnecessary complication in our paper, we choose to define $\underline{u}$ always as an element with lower oscillation k, independent of the parity of k.

(2) The condition that T be bounded can be eliminated by appropriately defining $u \downarrow h_i : -\infty$ and $u \uparrow h_i : +\infty$, and allowing $\pm\infty$ as "contributing" points in the oscillation sequence. However, the same effect is achieved by contracting T to a bounded set (say, by $u \to u \circ \tan^{-1}$ for $u \in \mathscr{F}(T)$ and $u \circ \tan^{-1} \in \mathscr{F}(\tan^{-1}(T))$), finding $\underline{u}$ and $\bar{u}$ in the new space, and then mapping back to the original space.

(3) Suppose a T-space $U \subset \mathscr{F}(T)$ has a basis $\{u_i\}_{i=0}^{k}$. Then multiplication of each element of U by $V(t) = 1/\max\{|u_0(t)|, |u_1(t)|, \ldots, |u_k(t)|\}$ gives a new T-space of bounded functions. However, an element of oscillation k in this new space does not necessarily pull back to an element of oscillation k in the original space. The reason is that asymptotic zeroes can be destroyed in the process.

(4) The conditions that the elements of U be bounded cannot in general be relaxed. For example, the T-space U of degree 1 spanned by 1, $\tan t$ on $]-\pi/2, \pi/2[$ satisfies $0 \in]]-1, 1[[$ and yet there is no element of $[-1, 1] \cap U$ of lower oscillation 1.

(5) We note that Corollary 5.3, which is almost Theorem 5.1, can be proved through more direct geometric means, exploiting only the properties of finite dimensionality, compactness, and convexity as they occur.

6. A Converse

From Theorem 5.1 we easily obtain

(6.1) Theorem. *Let T be a bounded subset of $\mathbb{R}$ and suppose $U \subset \mathscr{B}(T)$ is a T-space of degree k. Given any subset $S \subset T$ such that* card $S > k$ *and any two functions $h_0, h_1 \in \mathscr{F}(S)$ such that $]]h_0, h_1[[\cap U|_S \neq \emptyset$ there exist $\underline{u}$, $\bar{u} \in U$ such that the lower oscillation of $\underline{u}|_S$ and the upper oscillation of $\bar{u}|_S$ relative to $[h_0, h_1]$ are both equal to k.*

We show here that this theorem holds only for T-spaces. That is, we show

(6.2) THEOREM. *Let $T \subset \mathbb{R}$, such that* card $T > k$. *Let $U \subset \mathscr{F}(T)$ be a real vector space of dimension at most $k + 1$. If for each $S \subset T$ satisfying* card $S > k$ *and each $h_0, h_1 \in \mathscr{F}(S)$ such that $]]h_0, h_1[[\cap U|_S \neq \varnothing$, there exists a $\underline{u} \in U$ ($\bar{u} \in U$) such that $\underline{Q}(\underline{u}|_S) = k(\bar{O}(\bar{u}|_S) = k)$ relative to $[h_0, h_1]$, then U is a T space of degree k.*

Proof. Consider any set of points $S = \{\tau_0 < \tau_1 < \cdots < \tau_k\} \subset T$. Let $h_0, h_1 \in \mathscr{F}(S)$ be such that $0 \in]]h_0, h_1[[$. Hence, by the assumption of the theorem there is a $\underline{u} \in U$ such that $\underline{Q}(\underline{u}|_S)$ relative to $[h_0, h_1]$ is k. Therefore

$$\underline{u}(\tau_i) = h_i(\tau_i) \tag{1}$$

(here $h_i = h_0$) if i is even and $h_i = h_1$ if i is odd).

Let $u_0, u_1, \ldots, u_m$ be a basis for u where $m \leqslant k$ by assumption of the theorem. Then (1) implies that there exist c_i, $i = 0, 1, \ldots, m$, such that

$$\sum_{j=0}^{m} c_j u_j(\tau_i) = h_i(\tau_i).$$

Now it is easy to see that there exist $k + 1$ pairs of functions $h_0^j, h_1^j \in \mathscr{F}(S)$ such that $0 \in]]h_0^j, h_1^j[[$ and the vectors $(h_0^j(\tau_0), h_1^j(\tau_1), \ldots, h_k^j(\tau_k))$ are a linearly independent set for $j = 0, 1, \ldots, k$. Hence the assumptions of the theorem imply that $m = k$ and if the matrix V is defined by $V_{ij} = u_j(\tau_i)$ $i, j = 0, 1, \ldots, k + 1$ then

$$\det V \neq 0. \tag{2}$$

Hence it follows that $Z(u) \leqslant k$ for any $0 \neq u \in U$.

We next show that U is a T-space by showing that $S^-(u) \leqslant k$ for every $u \neq 0$ in U (see [2]). Suppose there exists a $u \in U$ such that $S^-(u) \geqslant k + 1$. We may assume without loss of generality that there exist $k + 2$ points $t_i \in T$, $i = 0, 1, \ldots, (k + 1)$ such that $(-1)^i u(t_i) > 0$, $i = 0, 1, \ldots, k + 1$, and $t_{i-1} < t_i$, $i = 1, \ldots, k + 1$. Let

$$S' = \{t_0, t_1, \ldots, t_{k+1}\},$$
$$S = \{t_1, \ldots, t_{k+1}\}.$$

Define $h_0, h_1 \in \mathscr{F}(S)$ such that

$$h_0(t_i) = -|u(t_i)|, \qquad i = 1, 2, \ldots, k + 1, \tag{3}$$

$$h_1(t_i) = |u(t_i)|, \qquad i = 1, 2, \ldots, k + 1. \tag{4}$$

Since $V(t_1, \ldots, t_{k+1})$ is such that $\det V \neq 0$ from (2) and u_i, $i = 0, \ldots, k$

are bounded on S' it can be easily shown that there exists $M < +\infty$ such that for all $u \in U|_S$,

$$u|_S \in [h_0, h_1] \cap U|_S \Rightarrow |u(t_0)| < M. \tag{5}$$

Now let $h_0', h_1' \in \mathscr{F}(S')$ such that

$$\begin{aligned} h_0'(t_0) &= -M, \\ h_1'(t_0) &= u(t_0)/2, \\ \left.\begin{aligned} h_0'(t_i) &= h_0(t_i) \\ h_1'(t_i) &= h_1(t_i) \end{aligned}\right\} & \quad i = 1, 2,..., (k+1). \end{aligned} \tag{6}$$

We show that there cannot in this case (when $S^-(u) \geq k+1$) be any element $\underline{u} \in U|_{S'} \cap [h_0', h_1']$ such that $O(\underline{u})$ with respect to $[h_0', h_1']$ is k, thus contradicting the assumptions of the theorem.

Since $\underline{u} \in U|_{S'} \cap [h_0', h_1']$ and $S \subset S'$,

$$\underline{u}|_S \in [h_0, h_1] \cap U|_S;$$

hence from (5), $|\underline{u}(t_0)| < M$, whence from the definition of lower oscillation sequence and the fact that $h_0(t_0) = -M$ it follows that the lower oscillation sequence of $\underline{u}$ has to be

$$-\infty, t_1, t_2,..., t_{k+1}, \infty, \infty,... .$$

Therefore $\underline{u}(t_i) = h_i(t_i)$, $i = 1, 2,..., k+1$, from the definition of lower oscillation sequences. Hence from (3) and (4), $\underline{u}(t_i) = u(t_i)$, $i = 1, 2,..., k+1$. Therefore $(\underline{u} - u) \in U$ has $(k+1)$ zeroes t_i, $i = 1, 2,..., k+1$. This implies $\underline{u} - u = 0$, since we already saw that $Z(\underline{u} - u) \leq k$ if $\underline{u} - u \neq 0$. Hence $\underline{u}(t_0) = u(t_0)$. However, by (6), $\underline{u}(t_0) > h_1'(t_0)$. Hence $\underline{u} \notin [h_0', h_1']$. Therefore for every $u \neq 0$ and $u \in U$ we have

$$\max\{Z(u), S^-(u)\} \leq k;$$

therefore U is a T-space of degree k (see [2]).

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References

1. L. Fejér, Über trigonometrische Polynome, *J. Reine Angew. Math.* **146** (1915), 53–82.
2. B. Gopinath and R. P. Kurshan, Embedding an arbitrary function into a Tchebycheff space, *J. Approximation Theory* **21** (1977), 126–142.

3. B. Gopinath and R. P. Kurshan, The existence in T-spaces of functions with prescribed alternations, *J. Approximation Theory* **21** (1977), 143–150.
4. S. Karlin, Representation theorems for positive functions, *J. Math. Mech.* **12** (1963), 599–618.
5. S. Karlin and L. S. Shapley, Geometry of moment spaces, *Mem. Amer. Math. Soc.*, No. 12, 1953.
6. S. Karlin and W. J. Studden, "Tchebycheff Systems: With Applications in Analysis and Statistics," Wiley, New York, 1966.
7. A. Pinkus, Applications of representation theorems to problems of Tchebycheff approximation with constraints, to appear.
8. G. Pólya and G. Szegö, "Aufgaben and Lehrsätze aus der Analysis," Vol. II, Springer, Berlin, 1925.
9. G. Szegö, "Orthogonal Polynomials," rev. ed., American Mathematical Society Colloquium Publications, Vol. 23, Amer. Math. Soc., New York, 1959.